Dynamics of Oil Slicks

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The spread of an oil slick onto calm water is considered from a theoretical viewpoint. The equations of motion are derived for the gravity-inertial and gravity-viscous flow regimes (in which the spreading force of gravity is balanced by, respectively, inertia and viscous drag at the water surface). Approximate formulations for the viscous drag are suggested. The initial growth of the one-dimensional slick is described. For both one-dimensional and radial slicks, similarity solutions are obtained for the flow regimes which give adequate agreement with available experimental data. Approximate solutions are also given for the surface tension-viscous regime, in which the slick becomes so thin that surface tension rather than gravity is the primary spreading mechanism. Based on these solutions, a general picture of the slick development is presented.

Introduction

THE spread of oil over water has recently become of concern because of the devastating and obnoxious effects of marine oil pollution. Oil spills are the result of carelessness, accidents or convenience on the part of oil producers and transporters, and there is little hope that they can be eliminated entirely. Because a few spills involving on the order of 100,000 tons of crude oil have focused worldwide attention on the problem, research is now accelerating on ways to control the damage, identify the guilty parties, and prevent recurrences. The topic has been of interest for many centuries, however, since the ancients recognized that oil had a calming effect on wind-driven waves. In more modern times Benjamin Franklin among others conducted large scale experiments on lakes and ponds, resulting in varying degrees of deliberate oil pollution.

Because we do not wish to contribute further to the devastation of the environment, we have confined our attention to theoretical aspects of the problem. Professor Fay[1] and coworkers at MIT have studied the spread of oil slicks over undisturbed water. From an order-of-magnitude estimate Fay deduced that three distinct flow regimes exist for the spread of oil on a calm ocean. At early times for a slick of sufficient size, the spreading rate is controlled by gravity and inertial forces. At later times (or for thinner slicks) the rate of spread depends on a balance of gravity and viscous forces acting on the oil-water interface. At very late times (or for very thin slicks), the further spread is forced by surface tension and retarded by viscous forces. Fay's functional forms for the spreading in the gravity-inertial and gravity-viscous regimes have been verified in recent experiments by Hoult, et. al.,[2] and in the surface tension-viscous regime by Lee.[3] Attempts at predicting the observed behavior on a purely theoretical basis (by Hoult and Suchon[4]) appear to have been somewhat unsuccessful. A comparison of these studies with those presented herein is included.

The theoretical considerations to follow cover all three regimes: gravity-inertial, gravity-viscous, and surface tension-viscous. We will treat both one-dimensional slicks, for which experimental information is available, and radial slicks. A thin layer integral approach is used in order to allow a purely analytic description.

Equations of Motion and Relation of Terms

In the first two flow regimes considered, three nondimensional parameters appear, the ratio of oil and water densities (or rather $1-\rho_o/\rho_w$), the oil slick depth-to-length ratio δ_o/L_o, and (for the viscous force) the characteristic Reynolds number for the water boundary layer. The nature of the flow (e.g., gravity-inertial or gravity-viscous) must be determined by the relationship between these parameters.

The equations of motion in a form valid in both flow regimes are given below

$$\partial u/\partial x + \partial v/\partial y + ju/x = 0 \tag{1}$$

$$\partial u/\partial t + u\,\partial u/\partial x + v\,\partial u/\partial y = -(1/\rho_o)\,\partial p/\partial x + (\mu_o/\rho_o)\,\partial^2 u/\partial y^2 \tag{2}$$

$$\partial v/\partial t + u\,\partial v/\partial x + v\,\partial v/\partial y = -(1/\rho_o)\,\partial p/\partial y - g \tag{3}$$

Here $j=0$ for the planar case and $j=1$ for the radial case, with x taken to be the radial coordinate in that instance. The coordinate perpendicular to the water surface is y, and $y=0$ represents the horizontal water surface prior to the spill. We integrate Eqs. (1–3) across the oil layer of thickness δ from the oil-air interface (subscript e) to the oil-water interface (subscript i). In accord with the thin layer assumption we will neglect vertical accelerations. Equation (3) then yields the hydrostatic relation between the pressure in the layer and the oil thickness: $p - p_e = g\rho_o(y_e - y)$. By differentiating the pressure relation with respect to x and making use of the fact that the weight of the oil equals the weight of displaced water, we derive an expression for the pressure gradient, which is independent of y

$$\partial p/\partial x = g\rho_o[(\rho_w - \rho_o)/\rho_o]\,\partial\delta/\partial x = g\rho_o(\Delta\rho/\rho_w)\,\partial\delta/\partial x \tag{4}$$

It follows from Eq. (2), neglecting the small term $v\,\partial u/\partial y$ and the viscous term $(\mu_o/\rho_o)\,\partial^2 u/\partial y^2$, that u is independent of y. We shall in any case neglect $v\,\partial u/\partial y$ in this equation, but we note that even if the viscous term is retained, u is still very nearly independent of y. The reason for this is that the viscosity of crude oil is very much greater (by, say, one to two orders of magnitude) than that of water, so that the slick tends to move locally as a homogeneous slab relative to the water.

By integrating the momentum Eq. (2) across the layer, we can express the viscous term in a more convenient form

$$(\partial u/\partial t + u\,\partial u/\partial x)\,\delta = -g(\Delta\rho/\rho_w)\,\delta\,\partial\delta/\partial x - (\mu_o/\rho_o)\,(\partial u/\partial y)_i$$

Continuity of stress at the interface requires that $\mu_o(\partial u/\partial y)_i = \tau_w$, where $\tau_w = Cu^2 Re_w^{-1/2}$. The (water) Reynolds number Re_w is based on a suitably chosen reference velocity and the distance from the slick leading edge to the point considered. We then have

$$\partial u/\partial t + u\,\partial u/\partial x = -g(\Delta\rho/\rho_w)\,\partial\delta/\partial x - \tau_w/(\rho_o\delta) \tag{5}$$

The normal velocity difference $v_e - v_i$ is obtained from the continuity Eq. (1)

$$v_e - v_i = -\int_{y_i}^{y_e} (\partial u/\partial x + ju/x)\,dy = -\delta(\partial u/\partial x + ju/x)$$

The values of the normal velocity at the top and bottom of the layer v_e and v_i are given by

$$v_e = \partial y_e/\partial t + u\,\partial y_e/\partial x = (\Delta\rho/\rho_w)(\partial\delta/\partial t + u\,\partial\delta/\partial x)$$

$$-v_i = \partial y_i/\partial t + u\,\partial y_i/\partial x = (\rho_o/\rho_w)(\partial\delta/\partial t + u\,\partial\delta/\partial x)$$

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We then have

$$\partial\delta/\partial t + u\,\partial\delta/\partial x + \delta\,(\partial u/\partial x + ju/x) = 0 \qquad (6)$$

These equations govern the motion of an oil slick‡ and are subject to suitable conditions describing the mode of release of the oil. They have the same form as the equations governing tidal waves of finite amplitude[6] (the shallow water theory). The nonlinear terms $u\,\partial u/\partial x$, $u\,\partial\delta/\partial x$ cannot be neglected, being of the same order of magnitude as $\partial u/\partial t$, $\partial\delta/\partial t$.

To gain insight, we shall simplify the problem by assuming that the flow is planar (i.e., one-dimensional as opposed to radial), that the oil is released instantaneously, and that it initially has a rectangular shape. We take

$$\alpha = g\,\Delta\rho/\rho_w$$

At some time t_1 in the development of the slick, let δ_1 and L_1 denote a characteristic depth and length of the slick such that $L^2 = \delta_1 L_1$ is the area of the slick in the plane of motion. We can nondimensionalize the terms in the equations by setting $t = t_1 \bar{t}$, $x = L_1 \bar{x}$, $\delta = \delta_1 \bar{\delta}$, $u = L_1/t_1\, \bar{u}$, and

$$\tau_w = L_1 t_1^{-3/2} (\rho_w/\mu_w)^{1/2} \bar{\tau}_w$$

We then obtain

$$\partial\bar{\delta}/\partial\bar{t} + \bar{u}\,\partial\bar{\delta}/\partial\bar{x} + \bar{\delta}\,\partial\bar{u}/\partial\bar{x} = 0 \qquad (7)$$

$$\frac{\partial\bar{u}}{\partial\bar{t}} + \bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} = -\frac{\alpha\delta_1 t_1^{\,2}}{L_1^{\,2}}\frac{\partial\bar{\delta}}{\partial\bar{x}} - \frac{1}{\rho_o \delta_1}(\rho_w \mu_w t_1)^{1/2}\frac{\bar{\tau}_w}{\bar{\delta}} \qquad (8)$$

From Eq. (7), we see that all terms in the continuity equation remain of equal order. From Eq. (8), we find that the ratio of viscous to intertial terms is proportional to

$$(\rho_w \mu_w)^{1/2} L_1 t_1^{\,1/2} / (\rho_o L^2)$$

We can conclude that in the early stage of development of the slick, the viscous term in the momentum equation is much smaller than the inertial terms (both L_1 and t_1 small) and can therefore be neglected, leaving a balance between the inertial and gravity ($\alpha\,\partial\delta/\partial x$) terms. In the later stages, however, the viscous term comes to predominate over the inertial terms (both L_1 and t_1 large), so that the inertial terms on the left-hand side of Eq. (8) can then be neglected, and the viscous term is balanced by the gravity term. The crossover point between the gravity-inertial and gravity-viscous regimes would be expected to occur when the ratio of viscous to inertial terms becomes unity, or when $L_1 t_1^{\,1/2} = \rho_o L^2/(\rho_w \mu_w)^{1/2}$. If at that point the inertial and gravity terms are still approximately in balance, we would have for the crossover time $t_1 \sim \alpha^{-2/7} (\rho_o/\rho_w)^{6/7} (\rho_w/\mu_w)^{3/7} L^{8/7}$, or about 10 min for a crude-oil slick with $L^2 = 25\ \mathrm{m}^2$. It is therefore seen that the duration of the gravity-inertial regime is relatively short.

One-Dimensional Slicks: Initial Growth

We return to the dimensional form of the integral Eqs. (6) and (5), take $j = 0$, and neglect the viscous term.

We follow Lamb[6] and multiply Eq. (6) by an unknown function $f'(\delta)$, where the prime denotes differentiation with respect to δ. By adding the equation which results to Eq. (5), we obtain

$$(\partial/\partial t + u\,\partial/\partial x)[f(\delta) + u] = -\delta f'(\delta)(\partial/\partial x)[f(\delta) + u]$$

provided

$$\delta[f'(\delta)]^2 = \alpha$$

It follows that $f = 2c$, where

$$c = (\alpha\delta)^{1/2} \qquad (9)$$

is the wave velocity. We define the Riemann invariants

$$P = 2c + u$$
$$Q = 2c - u \qquad (10)$$

such that

$$\partial P/\partial t + (u + c)\,\partial P/\partial x = 0$$
$$\partial Q/\partial t + (u - c)\,\partial Q/\partial x = 0$$

Then P is constant for a geometrical point moving to the right with a velocity $u + c$ and Q is constant for a point moving to the left with a velocity $u - c$.

The problem of the initial growth of the slick is analogous to the water configuration after a dam separating two bodies of water has broken. The solution can be determined by the method of characteristics. The conditions at $t = 0$ for a slick which is initially flat and at rest are $u(x, 0) = 0$; $\delta(x, 0) = \delta_1$ for $0 \leqq x \leqq x_1$, $\delta(x, 0) = 0$ for $x > x_1$. Near the leading edge of the slick, we have from the characteristic equatons

$$P = 2c + u = 2c_1$$
$$Q = 2c - u = 0$$

Adding and subtracting, it follows that the thickness and velocity of the expanding slick near the leading edge are given by

$$c = \tfrac{1}{2}c_1 \qquad (11a)$$
$$\delta = \tfrac{1}{4}\delta_1 \qquad (11b)$$
$$u = c_1 = (\alpha\delta_1)^{1/2} \qquad (11c)$$

Accordingly, the initial spread of the slick is directly proportional to time: $x_{LE} = x_1 + c_1 t$. Equations (11) apply to a finite region of constant properties near the leading edge.

The receding edge of the disturbed region travels with the velocity $-c_1$ (see Stoker[7] for the details of this derivation). Between the receding edge and the region of constant thickness near the leading edge, there is an expansion region in which $P = 2c_1$. The straight characteristics within this region are given by

$$dx/dt = (x - x_1)/t = u - c = 2c_1 - 3c$$

The region is bounded on the leading-edge side by the line $dx/dt = u - c = \tfrac{1}{2}c_1$ (from Eqs. (11)), $x = x_1 + \tfrac{1}{2}c_1 t$. From the preceding considerations, we have for the distribution of thickness

$$\delta = \begin{cases} \delta_1 & (0 \leqq x \leqq x_1 - c_1 t) \\ \tfrac{1}{9}\delta_1[2 - (x - x_1)/(c_1 t)]^2 & (x_1 - c_1 t < x < x_1 + \tfrac{1}{2}c_1 t) \\ \tfrac{1}{4}\delta_1 & (x_1 + \tfrac{1}{2}c_1 t < x < x_1 + c_1 t) \end{cases} \qquad (12)$$

(see Fig. 1).

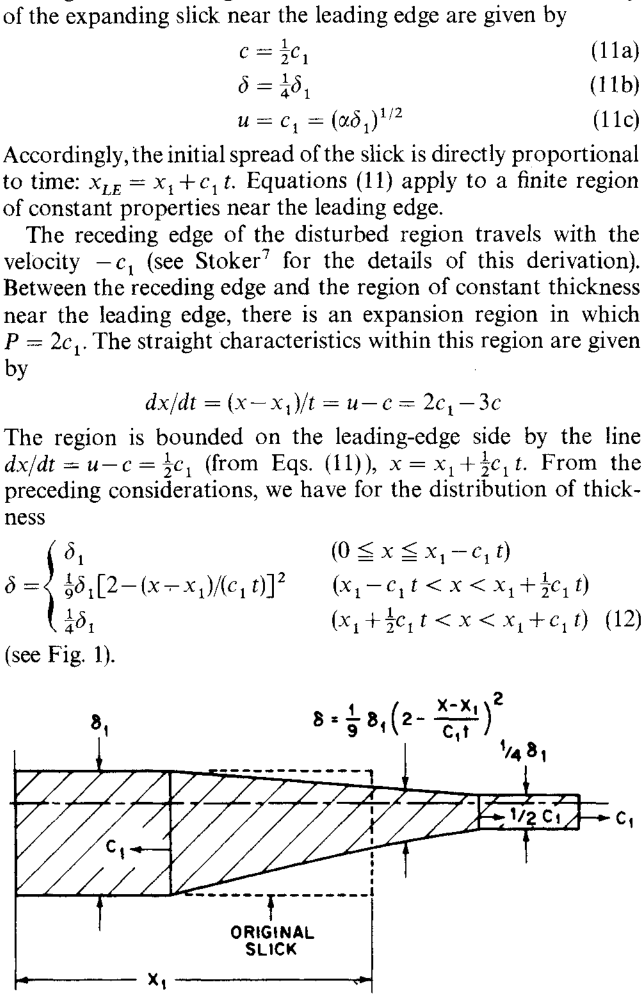

Fig. 1 Initial growth of slick.

The velocity of the leading edge in this solution is consistent with over-all mass conservation: the mass contained in the region bounded by the advancing and receding edges equals the mass originally contained in the region swept by the receding edge. It is interesting to note that the solution obtained by linearizing Eqs. (5) and (6) does not satisfy this over-all mass conservation requirement and hence must be rejected. In the corresponding acoustic problem, linearization is possible, but in that case the signal velocity greatly exceeds the bulk velocity in the fluid.

As a result of finite release rates and of reflections from the closed end, the ideal wave system discussed above will be somewhat different from that observed in experiments. In addition, the slick is not thin initially and thus may not be described adequately by our differential equations. Since the initial mode of

‡ The equations derived here correspond to Abbott's surface system[5] —i.e., to the limit in which the ratio of the slick thickness to the depth of water becomes vanishingly small. Abbott also derives the general equations which govern the coupled oil-water system when the oil thickness is not necessarily small in comparison to the water depth.

propagation (constant leading-edge velocity) has not been observed in experiments, it is appropriate to turn to the problem of slick propagation at later times, when a similarity solution can be derived.

Gravity-Inertial Regime

We shall look for a similarity solution applying to the motion at a distance from the point of release. We observe that such a solution has been derived for the one-dimensional case by Hoult and Suchon,[4] but with a constant of proportionality twice as large as the value indicated by experiments. Without going into the details of their solution, it is our belief that the discrepancy is due to their choice of the improper similarity variable x/t, which is not independent of the geometrical scale of the problem, rather than to any inadequacy of the experimental data.

In our analysis, we will solve the equations of motion subject to the conditions that the solution is similar in terms of the variable x/x_{LE}, that the total mass of the slick remains constant, that the velocity vanishes at the center, and that the velocity of the leading edge is equal to the characteristic wave speed c. We will present arguments to support the validity of the last condition.

Introducing the similarity variable

$$X = x/x_{LE}(t) \tag{13}$$

where

$$x_{LE} = At^n \tag{14}$$

Equations (6) and (5) become

$$\partial\delta/\partial t + (u/x_{LE} - nX/t)\,\partial\delta/\partial X + (\delta/x_{LE})(\partial u/\partial X + ju/X) = 0$$

$$\partial u/\partial t + (u/x_{LE} - nX/t)\,\partial u/\partial X + \alpha/x_{LE}\,\partial\delta/\partial X = 0$$

The equation of constancy of mass of the oil

$$(2\pi)^j \int_o^{x_{LE}} \delta x^j\,dx = L^{2+j} \tag{15}$$

where L^{2+j} is the initial volume of the spill, can only be satisfied in terms of similarity if

$$\delta = D(X)/x_{LE}{}^{1+j} \tag{16}$$

and we then have

$$\int_o^1 D(X)X^j\,dX = L^{2+j}/(2\pi)^j \tag{17}$$

In order to satisfy similarity, the velocity must have the form

$$u = U(X)x_{LE}/t \tag{18}$$

and the continuity equation can be written

$$(U'-n)D + (U-nX)D' + jD(U/X - n) = 0 \tag{19a}$$

or

$$(d/dX)\,[(U-nX)DX^j] = 0 \tag{19b}$$

$$(U-nX)DX^j = \text{const} \tag{19c}$$

The only solution to this equation satisfying the condition $U(0) = 0$ is

$$U = nX \tag{20}$$

Substituting into the momentum equation, integrating, and imposing Eq. (15), we obtain for the thickness distribution of the slick

$$D = (1+j)L^{2+j}\left[\frac{K^{3+j}}{(3+j)^2}X^2 + \frac{1}{(2\pi)^j} - \frac{1+j}{(3+j)^3}K^{3+j}\right] \tag{21}$$

where K is an integration constant, and

$$n = 2/(3+j)$$
$$A = K\alpha^{1/(3+j)}L^{(2+j)/(3+j)} \tag{22}$$

The value of the constant K must be determined by considering the rate of propagation of the blunt leading edge of the slick. By analogy with the problem of a bore in shallow water, the leading edge represents a finite disturbance with a velocity which depends on its thickness. For the late-time similar solution considered here, however, the disturbance can be taken to be infinitesimal.

Dimensional arguments lead us to the conclusion that the leading edge velocity must be proportional to the characteristic wave speed for a small disturbance $c = (\alpha\delta_{LE})^{1/2}$

$$u_{LE} = dx_{LE}/dt = k(\alpha\delta_{LE})^{1/2} \tag{23}$$

Theoretical determination of the constant k presents some difficulty due to the unknown interaction between the oil slick and the surrounding water near the leading edge. At late times, however, when the leading edge is very thin, the velocity would be expected to equal the wave speed ($k = 1$), by analogy with the acoustic limit in gasdynamics. This may be contrasted with the initial behavior of the leading edge, when according to Eqs. (11) $u_{LE} = c_1 = 2c$, giving $k = 2$. The values $k = 1$ and 2 represent lower and upper bounds n the possible range of variation of k. If we substitute Eq. (23) into Eqs. (13, 16, 21, and 22) evaluated at the leading edge, we obtain for the integration constant

$$K = \left[\frac{(3+j)^3(1+j)k^2}{2(2\pi)^j\{2(3+j)-(1+j)k^2\}}\right]^{1/(3+j)} \tag{24}$$

which ranges for one-dimensional flow ($j = 0$) from $\frac{3}{10}{}^{1/3} = 1.39$ ($k = 1$) to 3 ($k = 2$) (the latter, coincidentally, was the value obtained by Hoult and Suchon[4]). For radial flow ($j = 1$), K ranges from 1.14 ($k = 1$) to ∞ ($k = 2$).

Since our similarity solution is valid for late times, it is logical to adopt the late-time value $k = 1$ corresponding to an infinitesimal disturbance at the leading edge. Substantial empirical justification for this choice has been provided by Abbott,[5] whose experimental data are most satisfactorily correlated by the value $k = 1$. We then have the following formula for the spread of a one-dimensional slick

$$x_{LE}/L = 1.39(g\Delta\rho/\rho_w L)^{1/3}t^{2/3} \tag{25}$$

The coefficient 1.39 compares favorably with the value of 1.5 obtained by Hoult et al.[2] in their experimental work. For the radial case, we have

$$x_{LE}/L = 1.14(g\Delta\rho/\rho_w L)^{1/4}t^{1/2} \tag{26}$$

At the present time, no experimental data for radial slicks are available for comparison.§

This and the previous solution do not give constant energy in the slick; indeed, the energy can be seen to vary as $t^{-2(1+j)/(3+j)}$. Energy is presumed lost through turbulent dissipation near the blunt leading edge of the slick.

Gravity-Viscous Regime

When the slick becomes very long compared to its thickness, it is apparent from Eq. (8) that the acceleration terms can be neglected in the momentum equation, and Eq. (5) becomes

$$\alpha\delta\,\partial\delta/\partial x = -\tau_w/\rho_o \tag{27}$$

Before the simplified equations of motion can be solved, it is necessary to determine the drag exerted by the water on the slick. This can be accomplished either by solving the matched oil layer/water boundary-layer equations in detail, or by deriving a simplified model for the drag exerted by the water based on existing

§ It is reasonable to assume that at some later time the slick will become roughly flat, as it evolves from the initial behavior when it is thicker at the center to the late-time (similar) behavior when it is thicker at the leading edge. Assuming a flat slick, its thickness is given by $\delta = (1+j)L^{2+j}/(2\pi)^j$, and the wave speed is $dx_{LE}/dt = (\alpha\delta)^{1/2}$. Combining these expressions, we obtain a differential equation for x_{LE} which upon integration predicts the proper functional form for x_{LE} and gives for the coefficient K

$$K = [(3+j)/2]^{2/(3+j)}[(1+j)/(2\pi)^j]^{1/(3+j)}$$

This flat slick approximation is surprisingly accurate; the values $K = (\frac{3}{2})^{2/3} = 1.31$ ($j = 0$) and $K = 2^{1/2}/\pi^{1/4} = 0.94$ ($j = 1$) are quite close to the exact values derived above.

solutions. In Ref. 4 the former approach was adopted, but the authors were frustrated by mathematical difficulties. In this paper we have followed the latter course.

In our simplified drag model, we can make use of either the known Blasius boundary-layer solutions or solutions of the related Rayleigh problem. Neither method is exact or unique in its application, and various approximations can be made with a varying degree of accuracy in special regions. Unlike the analogous problem of boundary-layer growth in a shock tube, the present unsteady viscous problem cannot be reduced to one of steady flow by means of a Galilean transformation because the leading-edge speed varies with time. Available unsteady solutions are also difficult to apply since the speed varies from point to point (at fixed time) along the slick.

The simplest drag law is that for Rayleigh flow

$$\tau_w = (\mu \,\partial u/\partial y)_w = (\rho_w \mu_w/\pi)^{1/2} u_o/t_R^{\,1/2}$$

The problem now is to find relevant expressions for the reference velocity u_o and time t_R in terms of slick variables.

The average velocity is

$$\bar{u} = (1+j)\int_o^1 uX^j\,dX$$

An average time in motion $\bar{t}$ can be expressed as

$$\bar{t} = \tfrac{1}{2}x_{LE}/u_{LE}$$

where

$$u_{LE} = dx_{LE}/dt$$

If we base the drag law on average properties, we obtain

$$\tau_w = (2\rho_w \mu_w/\pi)^{1/2}\bar{u}(x_{LE}/u_{LE})^{-1/2} \tag{28}$$

If we approximate u by a straight line between $x = 0$ and $x = x_{LE}$, $\bar{u} = (1+j)u_{LE}/(2+j)$, and Eq. (28) becomes

$$\tau_w = [(1+j)/(2+j)](2\rho_w \mu_w/\pi)^{1/2} u_{LE}^{\,3/2} x_{LE}^{\,-1/2} \tag{29}$$

This is probably the simplest drag law that can be sensibly employed.

We can reason as an improvement that the region near the leading edge is most affected by viscous forces, and tailor our time in motion t_R to be as correct as possible in this region

$$t_R = (x_{LE} - x)/u_{LE}$$

We also use the local velocity rather than an average velocity $u_o = u$. We then obtain

$$\tau_w = \left(\frac{\rho_w \mu_w}{\pi}\right)^{1/2} u\, u_{LE}^{\,1/2}(x_{LE} - x)^{-1/2} \tag{30}$$

Equation (30) corresponds to local flat-plate similarity in boundary-layer theory. It has the appropriate singularity at the leading edge, and its form is such that a similarity-type solution appears possible. Because Eq. (30) represents a superior description of the drag variation, we will use it in the latter part of this section as our best drag law.

If we employ the average drag law Eq. (29), the x-momentum equation can be integrated directly. We will assume $x_{LE} = At^n$, where A and n are unknown constants. Substituting for x_{LE} and $u_{LE} = dx_{LE}/dt$ in Eqs. (27) and (29) and integrating, we obtain

$$\delta^2(x,t) - \delta^2(x_{LE},t) = \frac{2}{\alpha\rho_o}\left(\frac{2\rho_w \mu_w}{\pi}\right)^{1/2}\left(\frac{1+j}{2+j}\right) n^{3/2} A t^{n-3/2} \times (x_{LE} - x) \tag{31}$$

It is apparent from the expression above and from the physics of the problem that the maximum thickness is at the center and the minimum thickness occurs at the leading edge. In view of the fact that the problem considered has two unknowns (i.e., A and δ_{LE}) and only one condition (global mass conservation) which can be imposed, it becomes necessary to make an additional assumption. We will assume here that the slick thickness vanishes at the leading edge. This is both physically reasonable and consistent with experimentally observed behavior. We also note that the spreading laws derived for vanishing, and nonzero but small, leading-edge thicknesses would in fact differ very little. (The assumption $\delta_{LE} = 0$ may slightly overestimate the driving force $\alpha\,\partial\delta/\partial x$, but this is of little consequence in practical terms since the retarding force cannot be defined with absolute precision.)

Taking $\delta(x_{LE}, t) = 0$ in Eq. (31) and invoking the mass conservation condition Eq. (15), we obtain

$$n = 3/[4(2+j)] \tag{32}$$

$$A = \left[\frac{3\pi}{2}\frac{(2+j)^5}{(1+j)^2}\right]^{1/4(2+j)}\left(\frac{5}{4\pi}\right)^{j/(2+j)}(\alpha\rho_o)^{1/2(2+j)}(\rho_w \mu_w)^{-1/4(2+j)} L \tag{33}$$

We then have for the spread of a one-dimensional slick

$$x_{LE}/L = 2^{1/2}(3\pi)^{1/8}(\alpha\rho_o)^{1/4}(\rho_w \mu_w)^{-1/8} t^{3/8} \approx 1.87[g^2(\Delta\rho)^2/\rho_w \mu_w]^{1/8} t^{3/8} \quad \text{for} \quad \rho_o \approx \rho_w \tag{34}$$

According to Hoult et al.,[2] the experiments indicate a value for the coefficient of 1.5, which compares reasonably well with our value of 1.87. If we had based our drag law on the Blasius skin-friction coefficient rather than the corresponding Rayleigh value, the theoretical coefficient aforementioned would be 1.74. Although the ratio of the predicted viscous forces ($\pi^{-1/2}/0.332$) is about 1.7, the spreading laws are nearly equal due to the ($\frac{1}{8}$)-power in Eq. (34).

For the radial case, we have

$$x_{LE}/L = (15/4\pi)^{1/3}(9\pi/8)^{1/12}(\alpha\rho_o)^{1/6}(\rho_w \mu_w)^{-1/12} t^{1/4} \approx 1.18[g^2(\Delta\rho)^2/\rho_w \mu_w]^{1/12} t^{1/4} \quad \text{for} \quad \rho_o \approx \rho_w \tag{35}$$

Let us now consider an improved analysis based on the best drag law Eq. (30). Equations (6) and (27) are now coupled through the presence of the local velocity u on the right-hand side of Eq. (27). We will look for a similarity solution such as developed in the previous section. We introduce the similarity variable X as defined in Eq. (13), and observe that the treatment of the continuity equation leading to Eq. (20) is equally applicable here. Hence we can substitute Eqs. (16, 18, and 20) in Eqs. (27) and (30), obtaining Eq. (32) and the following differential equation for the thickness distribution

$$DD' = [3/4(2+j)]^{3/2}(\rho_w \mu_w/\pi)^{1/2}(1/\alpha\rho_o)A^{2(2+j)}X(1-X)^{-1/2} \tag{36}$$

Integrating Eq. (36) and imposing the condition $D(1) = 0$ discussed previously and the mass conservation condition Eq. (15), we obtain

$$D = [L^{2+j}/(2\pi)^j I](2+X)^{1/2}(1-X)^{1/4} \tag{37}$$

where

$$I = \int_o^1 X^j(2+X)^{1/2}(1-X)^{1/4}\,dX \tag{38}$$

and

$$A = \left[\frac{(\pi/3)^{1/4}2^{1/2}(2+j)^{3/4}}{(2\pi)^j I}\right]^{1/(2+j)}(\alpha\rho_o)^{1/2(2+j)}(\rho_w \mu_w)^{-1/4(2+j)} L \tag{39}$$

For the one-dimensional case, this gives

$$x_{LE}/L = 1.39(\alpha\rho_o)^{1/4}(\rho_w \mu_w)^{-1/8} t^{3/8} \approx 1.39[g^2(\Delta\rho)^2/\rho_w \mu_w]^{1/8} t^{3/8} \quad \text{for} \quad \rho_o \approx \rho_w \tag{40}$$

with the coefficient in good agreement with the experimental value[2] of 1.5. For the radial case, we have

$$x_{LE}/L = 0.98(\alpha\rho_o)^{1/6}(\rho_w \mu_w)^{-1/12} t^{1/4} \approx 0.98[g^2(\Delta\rho)^2/\rho_w \mu_w]^{1/12} t^{1/4} \quad \text{for} \quad \rho_o \approx \rho_w \tag{41}$$

As with the gravity-inertial regime, no experimental data are available for comparison with the radial result.

The results Eqs. (40) and (41) indicate a somewhat slower rate of spread than that calculated with the simpler drag law. This may be explained by noting that while the simpler drag law gives constant drag from the center to the edge at any instant of time, the best drag law, Eq. (30), vanishes at the center, is infinite at the edge, and matches the value of the simpler drag law at the midpoint; hence more drag is concentrated near the edge. Both solutions give an infinite slope at the leading edge.

Surface Tension-Viscous Regime

At very late times (or for very thin slicks) the dominant spreading force is the net difference of the surface tensions

Table 1 Predicted spreading laws for oil slicks

	Gravity-inertial regime	Gravity-viscous regime	Surface tension-viscous regime
One-dimensional	$x_{LE}/L = 1.39(g\Delta\rho/\rho_w L)^{1/3}t^{2/3}$ (1.5 from Ref. 2)	$x_{LE}/L = 1.39[g^2(\Delta\rho)^2/\rho_w\mu_w]^{1/8}t^{3/8}$ (1.5 from Ref. 2)	$x_{LE} = 1.43[\sigma^{1/2}/(\rho_w\mu_w)^{1/4}]t^{3/4}$ (1.33 from Ref. 3)
Radial	$x_{LE}/L = 1.14(g\Delta\rho/\rho_w L)^{1/4}t^{1/2}$	$x_{LE}/L = 0.98[g^2(\Delta\rho)^2/\rho_w\mu_w]^{1/12}t^{1/4}$	$x_{LE} = 1.6[\sigma^{1/2}/(\rho_w\mu_w)^{1/4}]t^{3/4}$

between; a) the oil-air and oil-water interfaces and b) the water-air interface. This difference is denoted by the spreading coefficient σ and its value must be determined from physical measurements. The dominant retarding force is again the viscous drag of the water beneath the slick.

An exact analysis of this flow problem appears to be a complex task. In what follows we will present a simple engineering approximation. This allows us first of all to correlate existing experimental data for one-dimensional flows,[3] but also to make reliable predictions for axisymmetric and other situations of practical interest for which no experimental data exist. We note that the rate of spread in the surface-tension driven regime is independent of the volume of the spill (see Fay[1]), and we would therefore expect the different configurations (one-dimensional or radial) to spread at nearly the same rate.

The force balance for the one-dimensional and radial cases can be expressed as follows

$$(2\pi x_{LE})^j\sigma = \int_o^{x_{LE}} (2\pi x)^j \tau_w\, dx \tag{42}$$

An equation for the unknown position of the leading-edge x_{LE} is obtained by substituting the expression for τ_w from our best drag law, Eq. (30). By letting $x_{LE} = At^n$ we find that dimensional consistency requires $n = \frac{3}{4}$, in agreement with experimental observations.[3] We take $u = u_{LE}(x/x_{LE})$ (this is consistent with all the previous solutions). Substituting and integrating Eq. (42), we find for the one-dimensional case

$$x_{LE} = (4\pi/3)^{1/4}[\sigma^{1/2}/(\rho_w\mu_w)^{1/4}]t^{3/4} \tag{43}$$

The coefficient $(4\pi/3)^{1/4} = 1.43$ is in good agreement with the experimental value 1.33 in Ref. 3. For the radial case we obtain

$$x_{LE} = 1.6[\sigma^{1/2}/(\rho_w\mu_w)^{1/4}]t^{3/4} \tag{44}$$

almost the same result as for the one-dimensional slick.

Summary of Results

Similarity solutions have been derived for the one-dimensional and radial spread of oil slicks over calm waters. Such solutions are valid for times sufficiently large that the initial configuration of the oil slick prior to its sudden release no longer affects the motion. Three regimes have been considered. In the first, the gravity-inertial regime, the viscous drag at the oil-water interface is negligible, and the equations of motion have the form of those governing tidal waves of finite amplitude, or shallow-water theory; this regime corresponds to the early period of growth of the slick. The similarity solution for this regime gives a slick which is thicker at the leading edge than at the center. Because the slick is assumed to be very thin, the propagation speed of the blunt leading edge is simply the characteristic wave speed corresponding to the leading-edge thickness.

In the second, gravity-viscous regime, the drag of the water on the slick is much larger than the inertial terms in the momentum equation; this regime corresponds to a later period in the slick growth, when the thickness of the slick is very small in comparison to that in the gravity-inertial regime. Approximate expressions for the drag are derived by analogy with the Blasius and Rayleigh boundary-layer solutions. In deriving solutions to the equations of motion for the slick in this regime, we take the leading edge to have zero thickness.

For completeness, an approximate solution for the surface tension-viscous regime has also been derived. The viscous drag has been specified by the same drag laws used in the gravity-viscous regime. For the case of surface-tension driven spread, the geometry effects are found to be unimportant.

Table 1 summarizes the spreading laws for the three regimes. In each case, the best solution is given, involving the most accurate description of viscous drag and the most sophisticated approach to a similarity solution. In those cases where experimental data are available from Refs. 2 and 3, the experimental constant is added in parenthesis. In Table 1, we have made the approximation $\rho_o \approx \rho_w$.

In addition, the initial growth of the slick has been determined for the one-dimensional case; the problem is similar to the problem of a dam breaking. Our findings for the one-dimensional slick are summarized in Fig. 2. Initially, a lamina moves out from the edge of the slick, joined to the contracting central part by a smooth transitional contour. The thickness of the advancing lamina is one-quarter the original thickness of the slick. The waveform predicted at the leading edge is square, but the retarding action of the water and the vertical acceleration of the oil in the vicinity of the wave, neglected in this analysis, would tend to round off the wave front. At some intermediate time, the slick is expected to be approximately flat. Later, the slick would tend to approach the parabolic form predicted by the similar solution for the gravity-inertial regime, with a larger part of the mass in the outer region. Although this solution predicts a vertical front for the slick, the retarding action of the water and the vertical acceleration again would tend to round off the leading edge. Still later, viscous drag comes to predominate over the inertial terms as the slick slows down, and most of the slick tends to assume the form predicted by the similarity solution for the gravity-viscous regime. The larger part of the mass is now found in the inner region of the slick. Since the density of oil differs typically from that of water by only about 20%, most of the slick lies below the water level. As the deceleration of the slick continues, the effect of gravity eventually falls off to zero and surface tension takes over as the driving force.

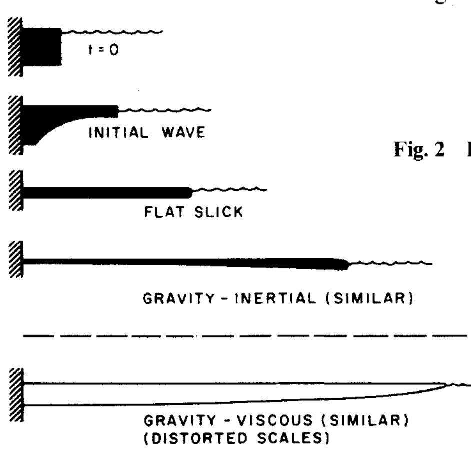

Fig. 2 Evolution of a slick.

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